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Paracategories I: internal paracategories and saturated partial algebras

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Abstract

Based on the monoid classifier Δ, we give an alternative axiomatization of Freyd's paracategories, which can be interpreted in any *bicategory of partial maps*. Assuming furthermore a free-monoid monad T in our ambient category, and coequalisers satisfying some exactness conditions, we give an abstract *envelope* construction, putting paramonoids (and paracategories) in the more general context of *partial algebras*. We introduce for the latter the crucial notion of *saturation*, which characterises those partial algebras which are isomorphic to the ones obtained from their enveloping algebras. We also set up a factorisation system for partial algebras, via epimorphisms and (monic) Kleene morphisms and relate the latter to saturation.

Keywords: Partial T-algebras; Monoid classifier; Paracetegories; Saturation

1. Introduction

There are two primary sources for this work: Freyd's proposal of *paracategories*, which arise by considering the restricted composition structure on an arbitrary subcollection of arrows of a category, and Mateus et al. [16,15] notion of *precategories* which provide a setting for probabilistic automata. Precategories are graphs endowed with identities and *binary* partial operation of composition, subject to suitable associative

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and unit laws. However, all the examples of precategories provided by the authors (and which we reexamine in [9]) are more naturally exhibited as paracategories, which provide a richer framework to study such structures.

The definition of paracategory in [5] is phrased in elementary terms, using many-sorted first-order Horn formulae of partial terms, with primitive binary relations $\succcurlyeq$ (if the left-hand side is defined so is the right-hand one and then they are equal) and $=$ (Kleene equality, $a=b$ iff $a\succcurlyeq b$ and $b\succcurlyeq a$ cf. Remark A.9). The motivating examples of paracategories are bivariant functors and dinatural transformations, and (the one-object example) the composition of untyped λ-terms in normal form.

The one-object case, a so-called *paramonoid*, consists of a set M and n-ary partial operations $\otimes_n : M^n \rightharpoonup M$ (for every n), suitably related, so as to be 'partially associative' and unitary (see Section 2 for precise details). Hence the major differences of a paramonoid with respect to a monoid are

- the *partial nature* of the operations,
- the weakened associativity axioms.

Categorical algebra obliterates the (somewhat artificial) distinction, traditional in universal algebra, between primitive and derived operations. Hence for an ordinary monoid, and more generally for a monoid in a monoidal category $\mathbb{C}$, the whole collection of its operations can be neatly grouped into a *strong monoidal functor* from Δ into $\mathbb{C}$, where Δ is the category of finite ordinals and monotone functions (see Section A.3 for details). Our first development is to give an axiomatization of paramonoids in these terms (see Definition 3.3), and later express a paracategory as a particular instance of this abstract notion of paramonoid (see Definition 3.7).

The next step is the generalisation to this context of Freyd's *envelope* construction, which allows us to consider any paramonoid as a subparamonoid of a monoid with a given subobject of its carrier. This construction forces us to consider some additional structure in our ambient category $\mathbb{B}$ (which so far was only required to have some pullbacks) to carry it out. Specifically, we must assume that $\mathbb{B}$ admits a free monoid monad, as well as coequalisers, with some exactness properties (see Section 4). Then, we can construct the adjunction between the category of paramonoids and that of monoids-with-distinguished-subobjects, Proposition 4.2, which provides in fact a representation theorem for the objects of the former.

With the above additional structure on $\mathbb{B}$, we observe that paramonoids are a special case of *partial* T-algebras. This is the technical core of the paper (Section 5): we introduce the category of partial algebras for a monad (as a special instance of the notion of lax algebra in Section 5.1), we formulate the crucial notion of *saturation* for partial algebras (Section 5.2) and we produce the *enveloping algebra* construction (Section 5.3). Our main result (Theorem 5.9) asserts that the enveloping algebra has the expected universal property, and characterises saturated partial algebras as those which can be recovered from their enveloping ones. We conclude our foray into partial algebras showing that the notion of Kleene morphism (taken from Freyd's original formulation of paracategories) makes sense at the partial algebra level, forming part of a factorisation system in the more general context of relational algebras (Section 5.4), and tie this notion up with saturation (Corollary 5.19).

2. Freyd's paracategories

We recall from [5] the elementary definitions of paramonoid and paracategory and their corresponding morphisms.

- A *paramonoid* consists of a set M and n-ary partial operations $\otimes_n : M^n \rightharpoonup M$, which we write indistinctly as $[\_]$ for any arity. These operations are subject to the following axioms:
  1. $\otimes_0 : M^0(=1) \rightharpoonup M$ is total
  2. $\otimes_1 = (id, id) : M \rightharpoonup M$
  3. If $[\vec{y}]$ is defined then $[\vec{x}[\vec{y}]\vec{z}] = [\vec{x}\vec{y}\vec{z}]$

  where the equality in the last axiom above is Kleene equality (if either side is defined so is the other and then they are equal).
- A *paracategory* $\mathbb{C}$ consists of a directed graph $C_0 \overset{d}{\leftarrow} C_1 \overset{c}{\rightarrow} C_0$ and partial n-ary operations $\circ_n : C_n \rightharpoonup C_1$, where

  $$C_n = C_1 \times_{C_0} \cdots \times_{C_0} C_1 = \{(f_1, \ldots, f_n) | cf_i = df_{i+1}, 1 \leqslant i \leqslant n-1\}$$

  is the set of composable n-tuples of arrows. They are subject to the following axioms
  1. $\circ_0 = \iota : C_0 \rightharpoonup C_1$ is total. This yields *identity arrows* $id_A : A \to A$ for every object $A \in C_0$.
  2. $\circ_1 = (id, id) : C_1 \to C_1$
  3. If $\circ_n \vec{y}$ is defined, then

     $$\circ_{m+1+n}(\vec{x}, \circ_k \vec{y}, \vec{z}) = \circ_{m+k+n}(\vec{x}, \vec{y}, \vec{z}),$$

     where $\vec{x} \in C_m$, $\vec{y} \in C_k$ and $\vec{z} \in C_n$.
- A *functor* between paracategories $\mathbb{C}$ and $\mathbb{D}$ is a morphism of graphs

  $$(f_0 : C_0 \to D_0, f_1 : C_1 \to D_1)$$

  such that if $[\vec{x}]$ is defined, then $f_1[\vec{x}] = [\vec{f_1 x}]$ (notice that this entails preservation of identities). The functor is called a *Kleene functor* if $[\vec{f_1 x}] = f_1 y$ implies $[\vec{x}] = y$.
- A *subparacategory* of a paracategory $\mathbb{C}$ is a subgraph such that the inclusion is a Kleene functor.

We will show in Section 5.4 that Kleene functors form part of a factorisation system, in the more general context of partial algebras.

A category $\mathbb{C}$ and a subset of arrows $P \subseteq C_1$ (including the identity arrows) determines a subparacategory, to wit, that where $[\vec{x}]$ is defined if the composite of the tuple $\vec{x}$ in $\mathbb{C}$ belongs to P. Similarly a functor between categories $F : \mathbb{C} \to \mathbb{D}$ with distinguished subsets of arrows $P \subseteq C_1$ and $Q \subseteq D_1$ such that $f_1(P) \subseteq Q$ determines a functor between the induced paracategories (see Section 5 for a more general version of this construction in the setting of partial algebras for a monad). Let $\mathscr{P}ar\mathscr{C}at$ denote the category of paracategories and functors and $\mathscr{C}at_P$ the category of categories with distinguished subsets of arrows (containing the identities) and functors compatible with such subsets. The construction above yields a functor $U : \mathscr{C}at_P \to \mathscr{P}ar\mathscr{C}at$.

2.1. Proposition (Enveloping category [5]). *The functor* $U : \mathscr{C}at_P \to \mathscr{P}ar\mathscr{C}at$ *admits a fully faithful left adjoint* EC.

Proof. Given a paracategory $\mathbb{C}$ define a category $\mathsf{EC}(\mathbb{C})$ as follows: let $\mathsf{F}(\mathbb{C})$ be the free category on the underlying graph of $\mathbb{C}$ and let $\mathsf{EC}(\mathbb{C})$ be the quotient of $\mathsf{F}(\mathbb{C})$ by the relation

$$\langle x_1, \ldots, x_n \rangle \sim \langle [x_1, \ldots, x_n] \rangle$$

whenever $[x_1, \ldots, x_n]$ is defined. Such equivalence classes carry a naturally distinguished subset P, namely $\{\langle x \rangle/_\sim \mid x \in \mathbb{C}\}$. This construction extends in an obvious manner to functors of paracategories to yield the desired left adjoint. This left adjoint being fully faithful is equivalent to the unit $\eta : \mathbb{C} \to U\mathsf{EC}(\mathbb{C})$ being an isomorphism [14, Section IV.3, Theorem 1]. From $\mathsf{EC}(\mathbb{C})$ and P we recover a paracategory $\mathbb{C}'$ whose objects are the equivalence classes of singletons $\langle x \rangle/_\sim$ and whose partial composite is given by

$$[\langle x_1 \rangle/_\sim, \ldots, \langle x_n \rangle/_\sim] = \langle x_1, \ldots, x_n \rangle/_\sim$$

To conclude $\mathbb{C} \cong \mathbb{C}'$, we must establish that $\langle x_1, \ldots, x_n \rangle \bar{\sim} \langle y \rangle$ iff $[x_1, \ldots, x_n] = y$, where $\bar{\sim}$ is the 'congruence-closure' of $\sim$. This closure is formed by adding reflexivity, enforcing symmetry and transitivity, and closing it under composition. It is clearly the least-fixed point of a monotone operator $F_\sim$ on the lattice of relations, $\bar{\sim} = \mu F_\sim$ and by the finitary nature of this operator, the least fixed-point is computed as $\bigcup_{i=0}^{\omega} F^i$. Thus we can use induction on i to deduce that

$$(x_1, \ldots, x_n) F^i (y) \Rightarrow [x_1, \ldots, x_n] = y$$

the only interesting case being that where the pair is in F^i by closure under composition: let $\vec{x} = (\vec{l}\vec{x'}\vec{r}) F^i y$ by means of

- $\vec{x'} F^j y'$
- $(\vec{l}\ y'\ \vec{r}) F^k y$

for some $j, k < i$. By induction hypothesis, $[\vec{x'}] = y'$ and $[\vec{l}\ y'\ \vec{r}] = y$. By the third axiom for paracategories,

$$[\vec{l}\ \vec{x'}\ \vec{r}] = [\vec{l}\ y'\ \vec{r}] = y. \qquad \Box$$

One important consequence of the above enveloping category construction is that it explains precisely how every paracategory arises, namely by specifying a collection of arrows in a given category. Any paracategory can be recovered from its enveloping category.

2.2. Examples. Given the envelope construction, it is straightforward to present examples of paracategories as categories with a distinguished collection of arrows:

1. (From [5]) Let M be the monoid of β/η-equivalence classes of closed untyped lambda terms under the operation of composition (not application), given by the

combinator traditionally named by $B = \lambda xyz.x(yz)$ (hence the composition of x and y would be Bxy). Consider the subset N of (equivalence classes of) normal terms (that is, those on which the β-rule is not applicable). The composition of (representatives of) two such is not necessarily normal(isable), hence we get a paramonoid of equivalence classes of normal terms.

2. Consider categories $\mathbb{C}$ and $\mathbb{D}$ and the collection of bivariant functors $T : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$. We obtain a paracategory $\mathsf{DiNat}(\mathbb{C}, \mathbb{D})$ with such bivariant functors as objects and dinatural transformations ([14, Chapter IX, Section 4]) as morphisms. The (pointwise) composition of dinatural of transformations is not necessarily dinatural, hence the paracategory structure. Note that the ambient category is that of bivariant functors whose arrows are arbitrary collections of morphisms $\{\alpha_C : T(C,C) \to S(C,C)\}_{C\in\mathbb{C}}$ (no dinaturality required). We explore this example further in [9].
3. Consider a simply typed λ-calculus $\mathscr{L}$, and its collection of $\mathscr{S}et$ valued models $\mathsf{Mod}(\mathscr{L})$ (such a model is the same thing as a cartesian-closure-preserving-functor from the free ccc generated by $\mathscr{L}$ into $\mathscr{S}et$ [13]). For any pair of models M and N we have the notion of *logical relation* between them, which in pedestrian terms is a family of relations $R_\sigma : M(\sigma) \nrightarrow N(\sigma)$ indexed by the types σ, satisfying:

$$R_{\sigma\times\tau}(x,y) \;\Leftrightarrow\; R_\sigma(\pi x, \pi y) \wedge R_\tau(\pi' x, \pi' y)$$

$$R_{\sigma\Rightarrow\tau}(f,g) \;\Leftrightarrow\; \forall x : M(\sigma), y : N(\sigma).\quad R_\sigma(x,y) \Rightarrow R_\tau(fx, gy)$$

The usual set-theoretic composition of two such logical relations is not necessarily a logical relation. We thus obtain a paracategory whose objects are $\mathscr{S}et$-valued models of $\mathscr{L}$ and whose morphisms are logical relations between such models. As in the previous example, the ambient category has models as objects and type-indexed collections of relations as arrows (no logical-relation condition).

3. Internal paramonoids and paracategories

As we recall in Section A.3, the monoid classifier Δ (the category of finite ordinals and monotone maps) gives a neat way to organise the collection of n-ary operations of a monoid, so as to guarantee the relevant associativity conditions

$$\otimes_n(\otimes_{m_1}, \ldots, \otimes_{m_n}) = \otimes_{m_1+\cdots+m_n}.$$

This equation is forced by functoriality, because in Δ the ordinal [1] is a terminal object and hence there is only one morphism from $[(m_1 + \cdots + m_n)]$ into it. We take a similar approach to organise the n-ary operations associated to a paramonoid. The essential difference is that composites will not be preserved. We will require instead lax functoriality, so that

$$\otimes_n(\otimes_{m_1}, \ldots, \otimes_{m_n}) \leqslant \otimes_{m_1+\cdots+m_n}$$

as partial operations from $M^{m_1+\cdots+m_n}$ to M.

The approach outlined above has the immediate advantage of being internalisable in any bicategory of partial maps. Thus, we set out to internalise paramonoids as a laxified version of the notion of monoid, that is as ‘lax internal monoids in a category of partial maps’. In addition to laxity, there is a subtlety concerning condition (3) in the definition of a paramonoid, which leads us to introduce below the concept of *saturation* for a lax functor into the bicategory of partial maps.

As we recall in Section A.5, our ambient universe for paramonoids is a bicategory of partial maps: we consider a category $\mathbb{B}$ and a class of monos $\mathscr{M}$ in it with the appropriate closure conditions to make them suitable *domains of partial maps* and set-up the bicategory $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$.

Since the hom-categories in $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$ are mere preorders rather than general categories, this gadget is also referred to as a *locally ordered category*. Although we would work in this simplified context, we usually keep a more ‘constructive’ outlook on ‘proofs of entailment between predicates’ (those specifying the domains of partial maps) in the back of our minds, so that the general bicategorical framework is pervasively (albeit implicitly) present. Of course, the locally ordered situation allows us to simplify the exposition by dispensing with the coherence conditions pertaining to the structural 2-cells of a lax functor, but it should be clear that our definitions are appropriate for the more general case as well.

Let us reexamine condition (3) of Freyd’s definition of paramonoid: If $[\vec{y}]\downarrow$ then $[\vec{x}[\vec{y}]\vec{z}]=[\vec{x}\vec{y}\vec{z}]$. The Kleene equality amounts to the following two statements:
1. $[\vec{y}]\downarrow$ and $[\vec{x}[\vec{y}]\vec{z}]\downarrow$ imply both $[\vec{x}\vec{y}\vec{z}]\downarrow$ and $[\vec{x}[\vec{y}]\vec{z}]=[\vec{x}\vec{y}\vec{z}]$.
2. $[\vec{y}]\downarrow$ and $[\vec{x}\vec{y}\vec{z}]\downarrow$ imply $[\vec{x}[\vec{y}]\vec{z}]\downarrow$ (and $[\vec{x}[\vec{y}]\vec{z}]=[\vec{x}\vec{y}\vec{z}]$)

Bearing in mind that $\otimes_1 = id$ (by the second axiom of paramonoids), the first statement (1) is clearly the laxity condition

$$\otimes_{m+1+n}(\otimes_1^m, \otimes_k, \otimes_1^n) \leqslant \otimes_{m+k+n},$$

where $m=|\vec{x}|$, $k=|\vec{y}|$ and $n=|\vec{z}|$. The second statement (2) amounts then to the inclusion of domains

$$D_{m+k+n} \subseteq (\otimes_1^m \times \otimes_k \times \otimes_1^n)^{-1}(D_{m+1+n}) \cap (M^m \times D_k \times M^n),$$

where $D_n \subseteq M^n$ is the domain of definition of $\otimes_n$ and the inverse image is taken along the total maps of the operations. This second condition amounts thus to a certain *saturation* of the domains of the partial operations (it forces more definedness than that required by laxity alone). Its abstract counterpart at the level of lax functors into a bicategory of partial maps is the following. [1]

3.1. Definition (Saturated lax functor). Given a category $\mathbb{C}$ and a lax functor $F:\mathbb{C}\to \mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$ consider a pair of composable morphisms $f:X\to Y$ and $g:Y\to Z$ in $\mathbb{C}$ and

[1] See Section A.5 for notation.

the corresponding structural 2-cell $\delta_{f,g} : Fg \circ Ff \leqslant F(g \circ f)$ as displayed

$$\begin{tikzcd}
 & & P_{gf} \arrow[ddll, hook, "d_{gf}"'] \arrow[ddrr, "F(gf)"] & & \\
 & & Ff^*P_g \arrow[u, hook, "\delta"] \arrow[dl, hook, "Ff^*d_g"'] \arrow[dr] & & \\
 & P_f \arrow[dl, hook, "d_f"'] \arrow[dr, "Ff"] & & P_g \arrow[dl, hook, "d_g"'] \arrow[dr, "Fg"] & \\
FX & & FY & & FZ
\end{tikzcd}$$

The adjunction[2] $d_f \circ (\_) \dashv d_f^* : Sub(FX) \to Sub(P_f)$ induces by transposition a 2-cell (inclusion) $\hat{\delta}_{f,g} : Ff^*P_g \to d_f^* P_{gf}$.

The lax functor F is called *saturated* if, for any composable pair $\langle f, g\rangle$, the 2-cell $\hat{\delta}_{f,g}$ is an isomorphism.

More explicitly, the condition of saturation amounts to requiring that the square

$$\begin{tikzcd}
Ff^*P_g \arrow[r, hook, "\delta_{f,g}"] \arrow[d, hook, "Ff^*d_g"'] & P_{gf} \arrow[d, hook, "d_{gf}"] \\
P_f \arrow[r, hook, "d_f"'] & FX
\end{tikzcd}$$

be a pullback, and thus, an intersection of subobjects, as alluded in the analysis preceeding the definition.

3.2. Remark. We will further examine saturation in the more general context of *partial algebras* in Section 5.2, where we unveal its actual meaning as *descent data* (Proposition 5.5).

Recall from Section A.5 that $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$ has a monoidal structure given by the finite products of $\mathbb{B}$. We are finally in position to state our main definition:

3.3. Definition. Consider a category $\mathbb{B}$ with finite limits and a class of monos $\mathscr{M}$ as in Section A.5

- An *internal paramonoid* $\mathbf{M}$ in $\mathbb{B}$ is a *strong-monoidal saturated lax functor* $\mathbf{M} : \mathbf{\Delta} \to \mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$, such that the partial map

  $$F(¡ : \emptyset \to [1]) : \mathbf{M}(\emptyset) \rightharpoonup \mathbf{M}(\mathbf{1})$$

  is total.

[2] $Sub(X)$ denotes the preorder of subobjects of X.

- A *morphism of internal paramonoids* is a monoidal lax transformation with *total* components between the corresponding lax functors.

We have thus the category $\mathscr{P}ar\mathscr{M}on(\mathbb{B})$ of internal paramonoids in $\mathbb{B}$ (leaving the class $\mathscr{M}$ implicit).

Let us write $M = \mathbf{M(1)}$ for the underlying object of an internal paramonoid. We proceed to unravel the ingredients of the definition:

- The lax functor $\mathbf{M}$ being strong monoidal implies that it is determined on objects by M as $\mathbf{M}[n] \cong M^n$. This is because Δ is generated[3] by [1].
- The (unique) morphism $[n] \to [1]$ in Δ yields via $\mathbf{M}$ a partial morphism $\otimes_n : M^n \rightharpoonup M$, whose domain is an $\mathscr{M}$-subobject $d_n : D_n \hookrightarrow M^n$.
- Laxity of $\mathbf{M}$ amounts then to

$$\otimes_n(\otimes_{m_1}, \ldots, \otimes_{m_n}) \leqslant \otimes_{m_1+\cdots+m_n}.$$

To see this, notice that the left hand side is the image of the composite

$$[(m_1 + \cdots + m_n) - 1] \begin{array}{c}\to\\ \vdots\\ \to\end{array} [n-1] \to [1]$$

in Δ, the first factor being mapped to $\otimes_{m_1} \times \cdots \times \otimes_{m_n}$ because $\mathbf{M}$ is strong monoidal.
- Since $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$ is locally ordered and the identities (which are total) are maximal, $id \leqslant M\,id$ implies

$$M\,id = \otimes_1 = id$$

so that M is *normal* (and we get the second condition of the definition of paramonoid). The same observation applies to lax transformations between such functors.
- The requirement that the partial map $e = \mathbf{M}_{¡} : (\mathbf{M}\emptyset \cong)\mathbf{1} \to \mathbf{M}$ be total means that M has a constant e which, as we show below, behaves as an always-defined identity for the partial ‘multiplications’.
- Saturation captures the right-to-left direction of Freyd’s third axiom, namely the inclusion of domains

$$D_{m+k+n} \subseteq (\otimes_1^m \times \otimes_k \times \otimes_1^n)^{-1}(D_{m+1+n}) \cap (M^m \times D_k \times M^n)$$

we mentioned before the definition.

3.4. Remark. It is important to *not* require that $\mathbf{M} : \mathbf{\Delta} \to \mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$ preserve the local ordering of $\Delta([m],[n])$. Otherwise, as the second referee pointed out to us, composition would be total: $([xy],[z]) \leqslant ([x],[yz])$ and with $x = e$, the LHS is total and thus $[yz]$ would be always defined.

Notice that for a lax functor $\mathbf{M} : \mathbf{\Delta} \to \mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$, we have a partial unit $e : 1 \rightharpoonup M$ corresponding to the unique morphism $\emptyset \to [1]$.

[3] The status of Δ as a monoid classifier means that it is the *free monoidal category with an internal monoid.*

3.5. Proposition. *For a strong-monoidal saturated lax functor* $\mathbf{M}:\mathbf{\Delta}\to\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$, *if the unique morphism* $\text{¡}:M\to\mathbf{1}$ *is a strong-epimorphism,*[4] *the unit* $e:1\rightharpoonup M$ *is a total map.*

Proof. Laxity and saturation imply that $id=m\circ(id\otimes e)$, where m is the partial binary composition. Hence, the right-hand expression is total and Lemma A.10 implies that $id\otimes e$ is a total map as well, thus its domain map $id\times d$ is an isomorphism. The unique diagonal fill-in in

$$\begin{tikzcd}
M\times\mathbf{1} \arrow[r, "\pi"] \arrow[d, "(id\times d)^{-1}"'] & \mathbf{1} \arrow[dd, equal] \arrow[ddl, dashed] \\
M\times D(e) \arrow[d, "\pi"'] & \\
D(e) \arrow[r, hook, "d"'] & \mathbf{1}
\end{tikzcd}$$

shows that the domain of e, $d:D(e)\hookrightarrow\mathbf{1}$, is an isomorphism as well. $\square$

Thus we end up with a total unit e for the paramonoid as soon as we require that M be non-empty in an internal sense, i.e. that the unique morphism $!:M\to\mathbf{1}$ be a strong-epimorphism.

Next we examine the associativity of the various partial operations.

3.6. Proposition. *For a paramonoid* $\mathbf{M}:\mathbf{\Delta}\to\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$, *the following hold*:
1. $\otimes_n(x_1,\ldots,x_n)=\otimes_{n+1}(x_1,\ldots,e,\ldots,x_n)$
2. *If* $\otimes_k(x_1,.,x_k)$ *is defined then*

$$\otimes_{m+1+n}(y_1,.,y_m,\otimes_k(x_1,.,x_k),z_1,.,z_n)=\otimes_{m+k+n}(y_1,.,y_m,x_1,.,x_k,z_1,.,z_n).$$

Proof. (1)

$$\begin{aligned}\otimes_{n+1}(x_1,\ldots,e,\ldots,x_n) &\succcurlyeq \otimes_n(x_1,\ldots,x_n)\\ &= \otimes_n(x_1,\ldots,\otimes(x_i,e),\ldots,x_n)\\ &\succcurlyeq \otimes_{n+1}(x_1,\ldots,x_i,e,\ldots,x_n)\end{aligned}$$

where the $\succcurlyeq$ follow by laxity, while the equality holds by saturation.

(2) This is an instance of the saturation condition: the left-hand side is $\otimes_{m+1+k}\circ(\otimes_1^m,\otimes_k,\otimes_1^n)$. Laxity implies the left-to-right inequality. For the converse, we must see that the domains of definition of both sides agree. By saturation, the intersection of D_{m+k+n} with $M^m\times D_k\times M^n$ is the same as the preimage of D_{m+1+n} by $(\otimes_1^m,\otimes_k,\otimes_1^n)$,

[4] Orthogonal to monos (in $\mathscr{M}$). See [2] for this property in the context of factorisation systems.

which formulated with variables reads as

$$(y_1,.,y_m,x_1,.,x_k,z_1,.,z_n)\in D_{m+k+n}\wedge(x_1,.,x_k)\in D_k$$
$$\Downarrow$$
$$(y_1,.,y_m,\otimes_k(x_1,.,x_k),z_1,.,z_n)\in D_{m+1+n}. \qquad \square$$

Now we turn to the consideration of paracategories. Since our category $\mathbb{B}$ has finite limits, the same is true for $\mathbf{Spn}(\mathbb{B})(C,C)\equiv\mathbb{B}/C\times C$ and the class of monos $\mathscr{M}$ can be considered as a suitable class (*dominion*) for $\mathbf{Spn}(\mathbb{B})(C,C)$. Hence we can consider the bicategory of partial maps $\mathsf{Ptl}_{\mathscr{M}}(\mathbf{Spn}(\mathbb{B})(C,C))$. Notice also that a morphism $f:C\to D$ induces, by pullback, a *change-of-base* strong monoidal homomorphism of (monoidal) bicategories

$$f^*:\mathbf{Ptl}_{\mathscr{M}}(\mathbf{Spn}(\mathbb{B})(D,D))\to\mathsf{Ptl}_{\mathscr{M}}(\mathbf{Spn}(\mathbb{B})(C,C))$$

which takes a paramonoid $\mathbf{M}:\mathbf{\Delta}\to\mathsf{Ptl}_{\mathscr{M}}(\mathbf{Spn}(\mathbb{B})(\mathbf{D},\mathbf{D}))$ to a paramonoid $f^*\circ\mathbf{M}:\mathbf{\Delta}\to\mathsf{Ptl}_{\mathscr{M}}(\mathbf{Spn}(\mathbb{B})(\mathbf{C},\mathbf{C}))$ (composition with homomorphisms preserves saturation).

3.7. Definition. An *internal paracategory* in $\mathbb{B}$ with objects C_0 is an internal paramonoid C in $\mathbf{Spn}(\mathbb{B})(C_0,C_0)$. Given another internal paracategory D with objects D_0, a *parafunctor* between them is given by a morphism $f_0:C_0\to D_0$ and a morphism of internal paramonoids $f_1:C\to f_0^*(D)$.

Let us record the agreement of our definitions with the elementary ones:

3.8. Proposition. *In $\mathbb{B}=\mathscr{S}et$ with $\mathscr{M}$ all monos, the definitions of internal paramonoids and paracategories (Definitions 3.3 and 3.7) agree with Freyd's definitions in Section 2, and likewise for their morphisms.*

Proof. Proposition 3.6 shows that our definition of internal paramonoid/paracategory specialised to $\mathscr{S}et$ satisfies Freyd's axioms of Section 2. Conversely, Freyd's third axiom for paramonoid/paracategory implies both laxity (left-to-right direction of the Kleene equality) and saturation (right-to-left). $\square$

3.9. Remark. The above proposition shows that in a sense Freyd's formulation of paramonoid is more concise than the one we obtain via lax monoids. Yet, our definition reveals some latent structure in Freyd's definition and puts paramonoids/paracategories in the context of *partial algebras* (Section 5), where we have the right abstract ingredients to carry out the envelope construction and the representation of saturated partial algebras via their enveloping ones (Theorem 5.9). $\square$

4. The envelope of a paramonoid revisited

Our next step in the analysis of paramonoids/paracategories is to give an abstract version of Freyd's enveloping monoid construction, suitable for our internal treatment. We will reorganise the data of an internal paramonoid in a more concise form, namely as a *single* partial map, so as to carry out the construction of the enveloping monoid as a coequaliser. This links paramonoids to the partial algebras introduced in Section 5.

A major advantage of the definitions of internal paramonoid and paracategory (Definitions 3.3 and 3.7, respectively) is that they make sense in any category of partial maps $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$, with a fairly minimal amount of structure assumed, namely finite limits in $\mathbb{B}$ and the closure properties of the class $\mathscr{M}$ of monos.

We now need to assume further structure on $\mathbb{B}$. Namely, in order to internalise the construction of the enveloping monoid/category of Proposition 2.1, we assume that $\mathbb{B}$ admits the construction of *free monoids*. The usual formula for the free monoid in $\mathscr{S}et$ $\mathscr{S}et$,

$$TX = \coprod_n X^n$$

makes sense in any category $\mathbb{B}$ with finite products such that
1. $\mathbb{B}$ admits countable coproducts.
2. The functor $X \times \_ : \mathbb{B} \to \mathbb{B}$ preserves countable coproducts (for every object X).

4.1. Remark. There are other methods to construct free monoids. One is by taking the colimit of the ω-chain

$$0 \to S0 \to \cdots S^n 0 \to \cdots,$$

where $S = \mathbf{1} + (X \times \_) : \mathbb{B} \to \mathbb{B}$. Another alternative is to regard TX as an 'X-labelling' of $T\mathbf{1}$, taking advantage that the free monoid monad is cartesian, and hence determined by its value on the terminal object, $T\mathbf{1}$, which is the *natural numbers object*. The object $T\mathbf{1}$ embodies the combinatorics of the free construction; for an arbitrary object X, TX simply labels the combinatorial data in $T\mathbf{1}$ with data from X. This works for instance in an elementary topos with a natural numbers object (which may fail to have countable coproducts though). See [1] for a detailed account.

In order to construct the envelope of a paramonoid, in addition to the above conditions which yield free monoids, we assume that $\mathbb{B}$ admits coequalisers stable under pullbacks (thus $\mathbb{B}$ is regular cf. [3]). Let $\mathscr{M}on(\mathbb{B})_P$ be the category of monoids in $\mathbb{B}$ with a distinguished non-empty subobject (of their carriers) and morphisms preserving these, and let $U : \mathscr{M}on(\mathbb{B})_P \to \mathscr{P}ar\mathscr{M}on(\mathbb{B})$ be the forgetful functor: given a monoid M and a subobject $d : D \hookrightarrow M$, we get partial operations $\otimes_n : D^n \rightharpoonup D$ via the following pullback:

$$\begin{tikzcd}
\tilde{D}_n \arrow[rr, "\otimes_n"] \arrow[d, hook, "d_n"'] & & D \arrow[d, hook, "d"] \\
D^n \arrow[r, hook, "d^n"'] & M^n \arrow[r, "\otimes_n"'] & M
\end{tikzcd}$$

which organise themselves into a paramonoid structure on D. In $\mathscr{S}et$, $\tilde{D}_n = \{(m_1, \ldots, m_n) \in D^n \mid \otimes_n \ (m_1, \ldots, m_n) \in D\}$. Since the class $\mathscr{M}$ of domains of partial maps is stable under pullbacks, if $d \in \mathscr{M}$ then $d^n \in \mathscr{M}$. Recall that in any regular category (finite limits + pullback stable coequalisers), we can construct the image-factorisation of a morphism $f: X \to Y$ as follows

$$\begin{tikzcd}
 & & Im(f) \arrow[d, hook, "m"] \\
Ker_f \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & X \arrow[ur, dashed, "q"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

where $d, c: Ker_f \to X$ is the *kernel* of f, i.e. the pullback of f along itself, and $q: X \to Im(f)$ is their coequaliser. In $\mathscr{S}et$, $Ker_f = \{(x, y) \mid fx = fy\}$ and $Im(f) = X/Ker_f$. The resulting image functor $\Sigma_f: Sub(X) \to Sub(Y)$ is left-adjoint to the pullback functor $f^*: Sub(Y) \to Sub(X)$ between the categories of subobjects (thus, $Im(f) = \Sigma_f(id_X)$). Furthermore, such adjoints satisfy a stability condition, the so-called Beck–Chevalley condition, as a consequence of the pullback stability of coequalisers. A concise and extremely convenient embodiment of these facts is the following: consider the category $\mathsf{Sub}(\mathbb{B})$ whose objects are subobjects $d: D \hookrightarrow X$ and whose morphisms are commutative squares

$$\begin{tikzcd}
D \arrow[r, dashed] \arrow[d, hook, "m"'] & D' \arrow[d, hook, "m'"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}$$

and let $cod: \mathsf{Sub}(\mathbb{B}) \to \mathbb{B}$ be the evident forgetful functor, taking the codomain of the subobject. The fact that $\mathbb{B}$ is a regular category is equivalent to the statement that $\mathbb{B}$ has finite limits and cod is a fibration with sums (bifibration satisfying the Beck–Chevalley condition) and quotients. See [10] for a convenient account of fibred categorical matters (for the logically minded reader).

Since we are working with a distinguished class of monos $\mathscr{M}$, we would require a similar fibration-with-sums structure for $cod: \mathscr{M}(\mathbb{B}) \to \mathbb{B}$, with $\mathscr{M}(\mathbb{B})$ the evident full subcategory of $\mathsf{Sub}(\mathbb{B})$ spanned by the $\mathscr{M}$-subobjects. We thus have the following requirements of $\mathbb{B}$ with respect to $\mathscr{M}$:

1. $cod: \mathscr{M}(\mathbb{B}) \to \mathbb{B}$ is a fibration with sums.
2. The class $\mathscr{M}$ is stable under the relevant colimits (countable coproducts and coequalisers): given two diagrams $F, G: \mathbb{J} \to \mathbb{B}$ and a natural transformation $\alpha: F \Rightarrow G$ whose components are in $\mathscr{M}$, the induced morphism between the colimit objects $\varinjlim \alpha: \varinjlim F \to \varinjlim G$ is in $\mathscr{M}$.

Under these additional assumptions we have:

4.2. Proposition. *The functor $U: \mathscr{M}on(\mathbb{B})_P \to \mathscr{P}ar\mathscr{M}on(\mathbb{B})$ admits a left adjoint.*

Proof. We outline the construction, bearing in mind that this result is a special case of Theorem 5.9. Consider a paramonoid $\mathbf{M}: \mathbf{\Delta} \to \mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$. We organise all the partial operations

$$\otimes_n : M^n \rightharpoonup M = M^n \stackrel{d_n}{\hookleftarrow} D_n \stackrel{s_n}{\rightarrow} M$$

into a single span

$$TX \cong \coprod_n X^n \xleftarrow{\coprod_n d_n} \coprod_n D_n \stackrel{\langle s_n\rangle}{\rightarrow} M$$

and define the *enveloping monoid* $\mathsf{E}(M)$ by the following coequaliser

$$\begin{tikzcd}
T\coprod_n D_n \arrow[rr, "T\langle s_n\rangle"] \arrow[dr, "Td"'] & & TM \arrow[r, two heads, "q"'] & \mathsf{E}(M) \\
& {} \arrow[ur, "\mu"'] & &
\end{tikzcd}$$

where $\mu : T^2M \to TM$ is the multiplication of the free-monoid monad T and $d = \coprod_n d_n : \coprod_n D_n \hookrightarrow TX$. Our assumption on the stability of coequalisers under products(/pullbacks) implies that T preserves coequalisers, so the upper row of the following diagram is a coequaliser:

$$\begin{tikzcd}
T^2\coprod_n D_n \arrow[r, "T^2\coprod_n d_n"'] \arrow[rr, bend left, "T^2\langle s_n\rangle"] \arrow[d, "\mu"'] & T^3M \arrow[r, "T\mu"'] \arrow[d, "\mu T"'] & T^2M \arrow[r, two heads, "Tq"] \arrow[d, "\mu"] & T\mathsf{E}(M) \arrow[d, dashed, "s_{\mathsf{E}(M)}"] \\
T\coprod_n D_n \arrow[r, "T\coprod_n d_n"] \arrow[rr, bend right, "T\langle s_n\rangle"'] & T^2M \arrow[r, "\mu"] & TM \arrow[r, two heads, "q"'] & \mathsf{E}(M)
\end{tikzcd}$$

and all squares commute by naturality of μ and associativity for the monad T, hence the induced T-algebra structure on $\mathsf{E}(M)$. Finally, $\mathsf{E}(M)$ is endowed with an $\mathscr{M}$-subobject, namely the image $\Sigma_q(d)$. $\square$

5. Partial algebras and saturation

The construction of the envelope of a paramonoid in the proof of Proposition 4.2 above prompts us to consider the span

$$TM \xleftarrow{\coprod_n d_n} T\coprod_n D_n \stackrel{\langle s_n\rangle}{\rightarrow} M,$$

where $\{d_n : D_n \hookrightarrow M\}_n$ is the collection of domains of the partial operations of M. If the class of monos $\mathscr{M}$ is closed under coproducts (as it happens for ordinary monos in $\mathscr{S}et$), this span is itself a partial map. Thus, in the presence of the free-monoid monad T, the data for a paramonoid can be organised into a *single* partial map. The resulting structure is then an instance of the general notion of *partial algebra for a monad* which we proceed to examine.

5.1. Partial algebras for a monad

5.1. Definition. Given a category with finite limits $\mathbb{B}$ with a class of monos $\mathscr{M}$ satisfying the closure conditions of Section A.5 and a monad $\langle T, \eta, \mu\rangle$ such that $T(\mathscr{M}) \subseteq \mathscr{M}$, a *partial T-algebra* consists of an object X of $\mathbb{B}$, a partial map $x: TX \rightharpoonup X$ and a 2-cell

$$\begin{tikzcd} T^2X \arrow[r, "\mu"] \arrow[d, "Tx"'] \arrow[dr, phantom, "\overset{\alpha}{\leq}"] & TX \arrow[d, "x"] \\ TX \arrow[r, "x"'] & X \end{tikzcd}$$

satisfying the following *unit condition*:

$$x \circ \eta = id.$$

Notice that we get also $\alpha \circ T\eta = \alpha \circ \eta_T = id$ as there is only one way in which a partial map (x in this case) is $\leqslant$ to itself.

A morphism $f : x \to y$ between partial T-algebras $x: TX \rightharpoonup X$ and $y: TY \rightharpoonup Y$ is a morphism $f : X \to Y$ in $\mathbb{B}$ together with a 2-cell

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "x"'] \arrow[dr, phantom, "\leq"] & TY \arrow[d, "y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We have thus the category $\mathsf{Ptl}\text{-}T\text{-}\mathscr{A}lg$ of partial T-algebras and their morphisms.

A morphism of partial algebras $f : x \to y$ amounts thus to a tent

$$\begin{tikzcd} D_x \arrow[rrr, dashed] \arrow[dr, "x"] \arrow[dd, "d_x"'] & & & D_y \arrow[dr, "y"] \arrow[dd, "d_y"'] & \\ & X \arrow[rrr, "f"'] & & & Y \\ TX \arrow[rrr, "Tf"'] & & & TY & \end{tikzcd}$$

The *unit condition* amounts to the existence of a morphism $\eta'_x : X \to D_x$ such that

$$d_x \circ \eta'_x = \eta_X \quad x \circ \eta'_x = id$$

and the following being a pullback square

$$\begin{tikzcd} X \arrow[r, "\eta'_x"] \arrow[d, "id"'] & D \arrow[d, "d_x"] \\ X \arrow[r, "\eta_X"'] & TX \end{tikzcd}$$

5.2. Remark. Given $\mathbb{B}$ with finite limits and a class of monos $\mathscr{M}$, consider a monad $\langle T, \eta, \mu\rangle$ on $\mathbb{B}$. If the monad is *cartesian*, we get an induced monad $\mathsf{Ptl}_{\mathscr{M}}(T) : \mathsf{Ptl}_{\mathscr{M}}(\mathbb{B}) \to \mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$ (applying T to a partial map span yields another such) with unit (id, η) and multiplication (id, μ). The *lax algebras* for $\mathsf{Ptl}_{\mathscr{M}}(T)$, are the partial algebras introduced above. See [7] for relevant background material on these matters.

5.2. Saturation

In the case of paramonoids, we can recover them from their enveloping monoids. In the case of partial T-algebras, we would like to recover them from a corresponding *enveloping algebra* construction (see Definition 5.8 below). It would turn out that the partial algebras which could be thus obtained would be characterised (Theorem 5.9.(2) below) by the following crucial condition:

5.3. Definition. A partial T-algebra $TX \overset{d}{\leftarrow} D \overset{x}{\rightarrow} X$ is *saturated* when the adjoint transpose (across $Td \circ (\_) \dashv (Td)^* : Sub(T^2X) \to Sub(TD)$) of the structural 2-cell $\alpha : x \circ Tx \to x \circ \mu$, $\hat{\alpha} : Tx^* d \to Td^*(\mu^* d)$, is an isomorphism. More explicitly, the following square

$$\begin{tikzcd}
(Tx)^*D \arrow[r, hook, "\alpha"] \arrow[d, hook, "(Tx)^*d"'] & \mu^*D \arrow[d, hook, "\mu^*d"] \\
TD \arrow[r, hook, "Td"'] & T^2X
\end{tikzcd}$$

must be a pullback.

Recall that in the proof of Proposition 4.2 we came across the following essential data: the pair of morphisms $Tx, \mu \circ Td : TD \to TX$. Let us write $\bar{c} = \mu \circ Td$ and $\bar{d} = Tx$. There is further data associated to the graph $TX \overset{\bar{d}}{\leftarrow} TD \overset{\bar{c}}{\rightarrow} TX$:

- $T\eta'_x : TX \to TD$, which satisfies

$$\bar{d} \circ T\eta'_x = id, \quad \bar{c} \circ T\eta'_x = id.$$

- $T\alpha : TD \bullet TD \to TD$, where

$$\begin{tikzcd}[column sep=small]
 & & TD\bullet TD \arrow[dl, "\underline{\bar{d}}"'] \arrow[dr, "\underline{\bar{c}}"] & & \\
 & TD \arrow[dl, "\bar{d}"'] \arrow[dr, "\bar{c}"] & & TD \arrow[dl, "\bar{d}"'] \arrow[dr, "\bar{c}"] & \\
TX & & TX & & TX
\end{tikzcd}$$

and the square is a pullback (composition of spans) and $T\alpha$ is then a 2-cell between the corresponding spans.

5.4. Proposition (Internal category induced by a partial T-algebra).
- *The data* $\langle (TX, TD, \bar{d}, \bar{c}), T\eta'_X, T\alpha \rangle$ *forms an internal category in* $\mathbb{B}$.
- *This construction yields a functor* $\mathsf{cat}\colon \mathsf{Ptl}\text{-}T\text{-}\mathscr{A}lg \to \mathscr{C}at(\mathbb{B})$.

Recall that we have our fibration of $\mathscr{M}$-subobjects $cod\colon \mathscr{M}(\mathbb{B}) \to \mathbb{B}$. We have just seen that a partial algebra $x\colon TX \rightharpoonup X$ induces an internal category $\mathsf{cat}(x)$ in $\mathbb{B}$. Thus it makes sense to consider *descent data* for this internal category in the fibration *cod*. We are not going to make any substantial use of descent theory as to indulge in details, and simply refer the reader to [11,4]. The point is that the structural 2-cell α of the partial algebra renders the following diagram commutative:

$$\begin{tikzcd}
(Tx)^*D \arrow[r, "\alpha"'] \arrow[d, hook, "(Tx)^*d"'] \arrow[rr, bend left, "d^*(Tx)"] & \mu^*D \arrow[r, "d^*\mu"'] \arrow[d, hook, "\mu^*d"] & D \arrow[d, hook, "d"] \\
TD \arrow[r, "Td"] \arrow[rr, bend right, "Tx"'] & T^2X \arrow[r, "\mu"] & TX
\end{tikzcd}$$

which is an internal graph in $\mathscr{M}(\mathbb{B})$ (in fact, an internal discrete fibration) over the category $\mathsf{cat}(x)$. This commutative diagram induces the morphism $\hat{\alpha}\colon Tx^*d \to Td^*(\mu^*d)$ which in this context is better read as $\hat{\alpha}\colon \bar{d}^*d \to \bar{c}^*d$. Now we can give a different perspective on saturation:

5.5. Proposition. *The partial* T*-algebra is saturated iff the pair*

$$(d : D \hookrightarrow TX, \hat{\alpha} : \bar{c}^*d \to \bar{d}^*d)$$

constitute descent data for $cod\colon \mathscr{M}(\mathbb{B}) \to \mathbb{B}$ *over the internal category* $\mathsf{cat}(x)$.

Proof. The requirements for descent data are:
- $\hat{\alpha}$ must be an isomorphism (which is saturation)
- $\hat{\alpha}$ should satisfy the cocycle conditions, which are automatic in this case, since d (and all its pullbacks) is a monomorphism. □

As we will see in the proof of Theorem 5.9, this descent-theoretic perspective on saturation is helpful in establishing the main characteristic of saturated partial algebras, namely, that one such is isomorphic to the partial algebra induced by its enveloping algebra. As would be clear in the proof, we could also deduce the relevant properties from regularity of $\mathbb{B}$, but the above reformulation of saturation in terms of descent indicates what we should do if we worked constructively, keeping track of the 'proofs of entailments'.

5.3. Enveloping algebra

The construction of the internal enveloping monoid in the proof of Proposition 4.2 is already formulated at the level of partial T-algebras: it yields the free T-algebra on a partial one (plus a distinguished subobject of its carrier).

5.6. Definition. Let $T\text{-}\mathscr{A}lg_P$ be the category whose objects (x,P) are T-algebras $x: TX \to X$ together with a subobject of the carrier $m: P \hookrightarrow X$ and whose morphisms $f: (x,P) \to (y,Q)$ are T-algebra morphisms which preserve the subobjects:

$$\begin{tikzcd} P \arrow[r, dashed] \arrow[d, hook, "m"'] & Q \arrow[d, hook, "n"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

The functor $U: T\text{-}\mathscr{A}lg_P \to \mathsf{Ptl}\text{-}T\text{-}\mathscr{A}lg$ takes an algebra $x: TX \to X$ and a given $(\mathscr{M})$subobject $m: D \hookrightarrow X$ to the partial algebra given by the top span of the pullback

$$\begin{tikzcd} \tilde{D} \arrow[rr, "x^{\rightharpoonup}"] \arrow[d, hook, "d"'] & & D \arrow[d, hook, "m"] \\ TD \arrow[r, hook, "Tm"'] & TX \arrow[r, "x"'] & X \end{tikzcd}$$

The axioms for the partial algebra $(d, x^{\rightharpoonup}): TD \rightharpoonup D$ are easily verified:

- The unit axiom boils down to the fact that the pullback of a mono along itself is the identity span.
- The structural 2-cell
  $\alpha: (d, x^{\rightharpoonup}) \circ (Td, Tx^{\rightharpoonup}) \to (d, x^{\rightharpoonup}) \circ (id, \mu)$
  is induced (via functoriality of pullbacks) by the morphism of spans $Tm: (Tm, Tm) \to (id, id)$.

5.7. Corollary (Saturation of induced partial-algebras). *For any object* (x,P), *the induced partial algebra* $U(x,P)$ *is saturated.*

Proof. The corresponding structural 2-cell is constructed by pullbacks. □

5.8. Definition (Enveloping algebra). Assume the ambient category $\mathbb{B}$ has coequalisers and the monad functor T preserves them. Given a partial T-algebra

$$x: TX \rightharpoonup X = TX \xleftarrow{d_x} D \xrightarrow{x} X$$

define its *enveloping algebra* $\mathsf{E}(x)$ by the following coequaliser

$$\begin{tikzcd} TD \arrow[rr, shift left, "Tx"] \arrow[r, "Td_x"'] & \cdot \arrow[r, "\mu"'] & TX \arrow[r, two heads, "q"] & \mathsf{E}(x) \end{tikzcd}$$

with the corresponding algebra structure $s_x: T\mathsf{E}(x) \to \mathsf{E}(x)$ induced by the universal property of the coequaliser (preserved by T) as in the proof of Proposition 4.2. Likewise, we define a distinguished subobject on $\mathsf{E}(x)$ by taking the $\mathscr{M}$-image of d_x along q, $\Sigma_q(d_x): \bar{D} \hookrightarrow \mathsf{E}(x)$. Thus, the enveloping algebra yields a functor $\mathsf{E}: \mathsf{Ptl}\text{-}T\text{-}\mathscr{A}lg \to T\text{-}\mathscr{A}lg_P$.

Now we have all the ingredients to state our main result concerning partial algebras:

5.9. Theorem. *Assume* $\mathbb{B}$ *admits, and* T *preserves, coequalisers, and* $\mathscr{M}$ *is closed under images. The following hold*:
1. *The functors* E *and* U *are adjoint*:

$$\begin{tikzcd} \mathsf{Ptl}\text{-}T\text{-}\mathcal{A}lg \arrow[rr, bend left, "\mathsf{E}"] \arrow[rr, phantom, "\perp"] & & T\text{-}\mathcal{A}lg_P \arrow[ll, bend left, "U"] \end{tikzcd}$$

2. *For a partial algebra* $x : TX \rightharpoonup X$, *the unit of the adjunction* $\tilde{\eta}_x : x \to U\mathsf{E}(x)$ *is an isomorphism iff* $x : TX \rightharpoonup X$ *is saturated.*

Proof. (1) To verify that the enveloping algebra has the required universal property

$$\mathbf{Ptl}\text{-}T\text{-}\mathscr{A}lg(x, U(y,P)) \cong T\text{-}\mathscr{A}lg_P((\mathsf{E}(x), \Sigma_q(d_x)), (y,P))$$

consider an object (y,P) of $T\text{-}\mathscr{A}lg_P$, that is, an algebra $y : TY \to Y$ and a subobject $m : P \hookrightarrow Y$, and its associated partial algebra

$$\begin{tikzcd} \tilde{P} \arrow[rr, "y^{\rightharpoonup}"] \arrow[d, hook, "d"'] & & P \arrow[d, hook, "m"] \\ TP \arrow[r, hook, "Tm"'] & TY \arrow[r, "y"'] & Y \end{tikzcd}$$

and a morphism of partial algebras $f : X \to P$. We get the required unique morphism $\hat{f} : \mathsf{E}(x) \to Y$ as follows: the morphism $y \circ Tm \circ Tf : TX \to Y$ coequalises the pair $(Tx, \mu \circ Td_x) : TD \to TX$ and $\hat{f}$ is the induced factorisation through the coequaliser.

(2) We must spell out the unit of the adjunction $\tilde{\eta}_x : x \to U\mathsf{E}(x)$: we have a morphism $\bar{q} \circ \eta'_x : X \to \bar{D}$, where $\bar{q} : D \to \bar{D}$ is the instance of the unit of the adjunction $\Sigma_q \dashv q^*$ at the object d. More explicitly, consider the following diagram:

$$\begin{tikzcd} (Tx)^* D \arrow[r, "\alpha"'] \arrow[rr, bend left, "d^*(Tx)"] \arrow[d, hook, "(Tx)^* d"'] & \mu^* D \arrow[r, "d^*\mu"'] \arrow[d, "\mu^* d"'] & D \arrow[r, two heads, "\bar{q}"] \arrow[d, hook, "d"] & \bar{D} \arrow[d, hook, "\Sigma_q(d)"] \\ TD \arrow[r, "Td"] \arrow[rr, bend right, "Tx"'] & T^2 X \arrow[r, "\mu"] & TX \arrow[r, two heads, "q"'] & \mathsf{E}(x) \end{tikzcd}$$

A routine calculation verifies that the morphism $\bar{q} \circ \eta'_x : X \to \bar{D}$ is indeed a morphism of partial T-algebras, which is the unit $\tilde{\eta}_x$.

If $\tilde{\eta}_x$ is to be an isomorphism, Corollary 5.7 says that the partial algebra x must be saturated (saturation is an isomorphism-invariant property). It remains to show that saturation is sufficient. Appealing to Proposition 5.5, we descend along q by making the top row in the above diagram a coequaliser. Thus the image $\Sigma_q(d)$ is computed in this situation as the uniquely induced morphism between the coequalisers $\bar{q}$ and q. These considerations involving descent could also be deduced from the axioms of a regular category (specifically, coequalisers are pullback stable): the top part of the above diagram is a pullback of the bottom part (all squares are pullbacks).

To construct an inverse to $\tilde{\eta}_x$, consider the morphism $x : D \to X$. Since α is a morphism of partial maps, we have

$$x \circ d^*(Tx) = x \circ d^* \mu \circ \alpha$$

and by universality of the coequaliser $\bar{q}$, we get a unique factorisation of x through $\bar{q}$: there is $s : \bar{D} \to X$ such that $s \circ \bar{q} = x$. This is our purported inverse to $\tilde{\eta}_x$:

- $s \circ \tilde{\eta}_x = s \circ \bar{q} \circ \eta'_x = x \circ \eta'_x = id$
- $\tilde{\eta}_x \circ s = id$ since

$$\begin{aligned} \Sigma_q(d) \circ \tilde{\eta}_x \circ x &= q \circ \eta_X \circ x \\ &= q \circ Tx \circ \eta_D \\ &= q \circ \mu \circ Td \circ \eta_D \\ &= q \circ d \\ &= \Sigma_q(d) \circ \bar{q} \end{aligned}$$

  and $\Sigma_q(d)$ being mono implies

$$\tilde{\eta}_x \circ x = \tilde{\eta}_x \circ s \circ \bar{q} = \bar{q}$$

  which yields the desired equality since $\bar{q}$ is epi.
- We verify that $s : \bar{D} \to X$ is a morphism of partial algebras using the above mentioned fact that the square with the q's is a pullback and a routine calculation like the previous ones. □

The second assertion in the above theorem means that the saturation condition distinguishes those partial algebras which can be recovered from their enveloping algebra. In particular, when M is the free-monoid monad on $\mathscr{S}et$ and $\mathscr{M}$ is the class of all (regular) monos, we have our desired identification:

$$\boxed{\mathsf{Sat}^\bullet(\mathsf{Ptl} - \mathsf{M}\text{-}\mathscr{A}lg) \simeq \mathscr{P}ar\mathscr{M}on}$$

where $\mathsf{Sat}^\bullet(\mathsf{Ptl} - \mathsf{M}\text{-}\mathscr{A}lg)$ is the full subcategory of saturated partial M-algebras with non-empty carrier X ($! : X \to \mathbf{1}$ being a strong-epimorphism). Thus the only additional ingredient present in the case of paramonoids/paracategories is the assumption that the carrier is non-empty. Clearly if we impose a similar non-emptiness condition on subobjects in $T\text{-}\mathscr{A}lg_P$, Theorem 5.9 still applies, all the relevant constructions remaining unchanged.

5.10. Corollary (Reflectivity of saturated partial algebras). *The full subcategory* $\mathsf{Sat}(\mathsf{Ptl}\text{-}T\text{-}\mathscr{A}lg)$ *of saturated partial* T*-algebras is reflective, i.e. the inclusion* $\iota : \mathsf{Sat}(\mathsf{Ptl}\text{-}T\text{-}\mathscr{A}lg) \hookrightarrow \mathsf{Ptl}\text{-}T\text{-}\mathscr{A}lg$ *has a left adjoint* $\mathsf{Sat} : \mathsf{Ptl}\text{-}T\text{-}\mathscr{A}lg \to \mathsf{Sat}(\mathsf{Ptl}\text{-}T\text{-}\mathscr{A}lg)$.

Proof. Given an adjunction $F, G : \mathbb{D} \dashv \mathbb{C}$ with unit η such that ηG is an isomorphism, the full subcategory $\mathbb{C}'$ of $\mathbb{C}$ of those objects for which the unit is an isomorphism is reflective. Therefore the result follows from the characterisation of saturated partial algebras as those for which the unit of the adjunction of Theorem 5.9 is an isomorphism. In view of the developments to follow it is instructive to spell out the construction, assuming the unit $\eta : id \Rightarrow T$ is cartesian (see Section 5.4). Set $S(x) = U\mathsf{E}(x)$ and pull it back along $\tilde{\eta}_x$: if

$$\begin{tikzcd}
 & D_{\mathsf{Sat}(x)} \arrow[rr, dashed] \arrow[ddl, dashed, "d'"'] \arrow[dr, dashed, "x'"] & & D_{S(x)} \arrow[ddl, "d_{S(x)}"'] \arrow[dr] & \\
 & & X \arrow[rr, "\tilde{\eta}_x"'] & & \bar{D} \\
TX \arrow[rr, "T\tilde{\eta}_x"'] & & T\bar{D} & &
\end{tikzcd}$$

is a limit diagram, then $\mathsf{Sat}(x) = (d', x') : TX \rightharpoonup X$ is the required reflection. □

5.4. *Kleene morphisms and a factorisation system for partial algebras*

In the context of paracategories, Kleene functors play a role in defining the relevant notion of subparacategory (Section 2). Since they are only used in that context, it makes sense to consider the 'Kleene functor' condition only in the case when the underlying graph morphism is monic. We will show that such monic Kleene functors make sense at the level of partial algebras, where they form part of a factorisation system. We will also make clear at this level of generality Freyd's statement that the envelope construction embeds every paracategory as a subparacategory of a category. This will give yet another condition equivalent to saturation in Corollary 5.19.

The notion of monic Kleene morphism can be formulated at the level of partial algebras:

5.11. Definition. An $\mathscr{M}$-monomorphism $f : x \rightarrow y$ between partial T-algebras $x : TX \rightharpoonup X$ and $y : TY \rightharpoonup Y$ is a *Kleene morphism* if the corresponding tent diagram

$$\begin{tikzcd}
 & D_x \arrow[rr, dashed] \arrow[ddl, dashed, "d_x"'] \arrow[dr, dashed, "x"] & & D_y \arrow[ddl, "d_y"'] \arrow[dr, "y"] & \\
 & & X \arrow[rr, "f"'] & & Y \\
TX \arrow[rr, "Tf"'] & & TY & &
\end{tikzcd}$$

is a limit diagram, i.e. the dashed arrows constitute a (co)universal cone for the diagram given by the straight arrows.

In $\mathscr{S}et$-theoretic formulation:

$$\otimes^y(\vec{fy}) = fz \Rightarrow (\vec{y} \in D_x \wedge \otimes^x(\vec{y}) = z).$$

5.12. Proposition. *A functor* $(f_0, f_1)\colon \mathbb{C} \to \mathbb{D}$ *between paracategories with* $f_0 \in \mathscr{M}$ *is a Kleene functor iff the corresponding morphism of paramonoids* $f_1\colon C \to f_0^*(D)$ *is a Kleene morphism.*

5.13. Proposition (Fibrational property of partial algebras). *The forgetful functor* $\underline{U}$: $\mathsf{Ptl}\text{-}T\text{-}\mathscr{A}lg \to \mathbb{B}$, *which takes a partial algebra* $x\colon TX \rightharpoonup X$ *to* X, *admits cartesian liftings of* $\mathscr{M}$*-monomorphisms. An* $\mathscr{M}$*-monomorphism of partial algebras is cartesian iff it is a Kleene morphism. Furthermore, if the codomain of a cartesian morphism is saturated so is its domain.*

Proof. Given an $\mathscr{M}$-monomorphism $f\colon X \hookrightarrow Y$, we can construct the cartesian lifting of at a partial algebra $y\colon TY \rightharpoonup Y$ via the above limit diagram, which can be computed via pullbacks:

$$\begin{tikzcd}
 & & & \overline{\mathbb{D}} \arrow[dl, dashed, "\bar{f}"'] \arrow[dr, dashed] \arrow[ddll, bend right, "\bar{d}"'] \arrow[ddrr, bend left, "\bar{y}"] & & \\
 & & (Tf)^*D \arrow[dl, dashed, "(Tf)^*d"] \arrow[dr, dashed, "d^*(Tf)"] & & f^*D \arrow[dl, dashed, "y^*f"'] \arrow[dr, dashed, "f^*y"] & \\
 & TX \arrow[dr, "Tf"'] & & D \arrow[dl, "d"] \arrow[dr, "y"'] & & X \arrow[dl, "f"] \\
 & & TY & & Y &
\end{tikzcd}$$

Notice by the stability properties of $\mathscr{M}$ both $\bar{f}$ and $(Tf)^*d$ are in $\mathscr{M}$ and so is their composite $\bar{d}$. The resulting partial map $f^*y = (\bar{d}, \bar{y})\colon TX \rightharpoonup X$ is a partial algebra, the domain of the lifting.

As for universality, given any morphism of partial algebras $g\colon x \to y$ over $f\colon X \hookrightarrow Y$, we consider the corresponding factorisation through the limit diagram:

$$\begin{tikzcd}
D_x \arrow[r, hook] \arrow[drr, "x"'] \arrow[dd, "d_x"'] & f^*D_y \arrow[rr, dashed] \arrow[dr, dashed, "f^*x"] \arrow[ddl, dashed, "(Tf)^*d_y"] & & D_y \arrow[dr, "y"] \arrow[ddl, "d_y"'] & \\
 & & X \arrow[rr, "f"'] & & Y \\
TX \arrow[rr, "Tf"'] & & TY & &
\end{tikzcd}$$

which stabilises the required cartesian property. As for saturation, it is clear that the isomorphism required transfers via limits (pullbacks) from the codomain partial algebra to the domain one. $\square$

A morphism of partial algebras $f : x \to y$ is an *epimorphism* whenever $f : X \to Y$ is so (in $\mathbb{B}$). Recall from [2] the notion of factorisation system. As a consequence of Proposition 5.13 we have the following:

5.14. Proposition. *Any morphism $f : x \to y$ of relational algebras factors (uniquely) as an epimorphism followed by a (monic) Kleene morphism. The pair of classes of morphisms (epimorphisms/monic Kleene morphisms) constitutes a pullback stable factorisation system in* Ptl-T-$\mathscr{A}lg$.

Proof. Given a morphism of partial algebras $f : x \to y$, consider its underlying morphism $f : X \to Y$ in $\mathbb{B}$ and take its epi/$\mathscr{M}$-mono factorisation $f = m \circ e$ (the image factorisation of Section 4). Consider the cartesian lifting $\bar{m} : m^* y \to y$ and the corresponding induced map of partial algebras over e, $e' : x \to m^* y$ (which follows by cartesianness, or simply considering the limit diagram defining $m^* y$). This yields the required factorisation $f = \bar{m} \circ e'$. Both (epimorphisms) and (Kleene morphisms) are clearly stable under pullback. □

5.15. Remark. The reader familiar with factorisation systems in $\mathscr{C}at$ would recognise the above factorisation as formally analogous to the bijective-on-objects/fully-faithful factorisation of functors.

5.16. Remark. The bicategory of partial maps is a subbicategory of that of relations: the relational composition of two partial maps is again one such. Partial algebras are thus a special instance of *relational algebras*: a relational algebra for a monad $T : \mathbb{B} \to \mathbb{B}$ (which preserves jointly-monic spans) is given by a relation $x : TX \nrightarrow X$ and 2-cell $\alpha : x \circ Tx \to x \circ \mu$ satisfying $x \circ (id, \eta) = (id, id)$. Morphisms are defined as per partial algebras, yielding the category $\mathscr{R}el$-T-$\mathscr{A}lg$ of relational algebras. The category of partial algebras is thus a full subcategory of $\mathscr{R}el$-T-$\mathscr{A}lg$.

The forgetful functor $\underline{U} : \mathscr{R}el\text{-}T\text{-}\mathscr{A}lg \to \mathbb{B}$ is a fibration, whose cartesian liftings are precisely those morphisms such that the tent diagram is a limit. So the fibrational properties of Ptl-T-$\mathscr{A}lg$ of Proposition 5.13 are inherited from this ambient fibred category.

The construction in the proof of Corollary 5.10 is an instance of the cartesian lifting of $\mathscr{M}$-monos to partial algebras of Proposition 5.13. To see this, we must show that the unit $q \circ \eta_X : X \to \mathsf{E}(x)$ is a $\mathscr{M}$-monic.

5.17. Lemma. *If $\eta : id \Rightarrow T$ is a cartesian*[5] *transformation, that is, the corresponding naturality squares are pullbacks (or cartesian squares), whose instances are $\mathscr{M}$-monos, the unit of the adjunction of Theorem* 5.9 *is an $\mathscr{M}$-mono.*

[5] The property of being cartesian implies that the unit is monic. It usually holds for free-monoid constructions.

Proof. We would show that the composite $q \circ \eta_X$ is the $\mathscr{M}$-image of η_X along q.

Recall the definition of $\mathsf{E}(x)$ via the coequaliser

$$\begin{tikzcd} TD \arrow[rr, "Tx"] \arrow[dr, "Td_x"'] & & TX \arrow[r, two heads, "q"] & \mathsf{E}(x) \\ & {} \arrow[ur, "\mu"'] & & \end{tikzcd}$$

To compute the $\mathscr{M}$-image of (the $\mathscr{M}$-mono) η_X, we pull-back both morphisms defining the coequaliser along η_X, take their coequaliser, and obtain the image as the uniquely induced morphism between the coequalisers (cf. the proof of Theorem 5.9).

We show that pulling back the pair $(Tx, \mu \circ Td)$ along η_X yields a pair of equal morphisms, and thus its coequaliser is an isomorphism, and the corresponding image is the composite $q \circ \eta_X$ as desired. By cartesiannes of η, we have a pullback diagram

$$\begin{tikzcd} D \arrow[r, "x"] \arrow[d, "\eta_D"'] & X \arrow[d, "\eta_X"] \\ TD \arrow[r, "Tx"'] & TX \end{tikzcd}$$

For the other morphism, we have the pullback

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "\eta'_x"'] & X \arrow[r, "id"] \arrow[d, "\eta_X"'] & X \arrow[dd, "\eta_X"] \\ D \arrow[r, "d"] \arrow[d, "\eta_D"'] & TX \arrow[d, "\eta_{TX}"'] & \\ TD \arrow[r, "Td"'] & T^2X \arrow[r, "\mu"'] & TX \end{tikzcd}$$

where the top-left square is a pullback by the axiom for the partial algebra x (see Section 5.1), the bottom-left square is a pullback by cartesiannes of η, and the right rectangle is a pullback because η_X is mono, and thus in the following diagram

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "id"'] & X \arrow[r, "id"] \arrow[d, "\eta_X"'] & X \arrow[dd, "\eta_X"] \\ X \arrow[r, "\eta_X"] \arrow[d, "\eta_X"'] & TX \arrow[d, "\eta_{TX}"'] & \\ TX \arrow[r, "T\eta_X"'] & T^2X \arrow[r, "\mu"'] & TX \end{tikzcd}$$

the outer and top-left squares are pullbacks and the bottom-left square is a pullback by cartesianness of η. Since μ is (split) epi and we are in a regular setting, we conclude the right rectangle is a pullback indeed.

Thus we are left to coequalise the pair $(x \circ \eta'_x, id) : X \to X$ to obtain the image, but $x \circ \eta' = id$, again by the axiom for the partial algebra x. $\square$

5.18. Remark. In $\mathscr{S}et$, the composite $q \circ \eta_X$ being mono can be shown quite easily reasoning with elements:

$$(\langle z\rangle/_{\sim} = \langle y\rangle/_{\sim}) \Rightarrow \otimes_1\langle z\rangle = \otimes_1\langle y\rangle) \Rightarrow (z = y)$$

and we could have deduced the above result from this *logical argument* by appealing to the full exact embedding of a regular category in a Grothendieck topos cf. [3, Section 2.7].

Finally, let us relate Kleene morphisms with saturation of partial algebras. We have the composite adjunction

$$\begin{tikzcd} \mathsf{Ptl}\text{-}T\text{-}\mathcal{A}lg \arrow[r, bend left, "\mathsf{E}"] \arrow[r, phantom, "\perp"] & T\text{-}\mathcal{A}lg_P \arrow[l, bend left, "U"] \arrow[r, bend left, "\underline{U}"] \arrow[r, phantom, "\perp"] & T\text{-}\mathcal{A}lg \arrow[l, bend left, "\mathsf{Tot}"] \end{tikzcd}$$

where $\underline{U}$ forgets the distinguished subobject of the algebra and its right adjoint takes the algebra $x : TX \to X$ to the pair $(x, id : X \hookrightarrow X)$ (the *trivial* or *total* distinguished subobject). Let $\underline{T} = U\mathsf{Tot}\underline{U}\mathsf{E}$ the resulting monad on $\mathsf{Ptl}\text{-}T - \mathscr{A}lg$ with unit $\rho : id \Rightarrow \underline{T}$.

5.19. Corollary. *Under the hypothesis of Lemma* 5.17, *for a partial algebra* T*-algebra* $x : TX \rightharpoonup X$ *tfae*:
1. $x : TX \rightharpoonup X$ *is saturated*,
2. $\rho_x : x \to \underline{T}(x)$ *is a Kleene morphism*,
3. x *embeds into a* T*-algebra via a Kleene morphism.*

Proof. (1) $\Rightarrow$ (2): We only need to point out that the partial algebra induced by a total one $x : TX \to X$ and a given subobject $m : D \hookrightarrow X$ is the domain of the cartesian lifting of $\mathsf{Tot(x)}$ along the $\mathscr{M}$-mono m. The argument in the proof of Corollary 5.10 shows that when x is saturated the unit $\tilde{\eta}_x \cong \Sigma_q(d)$ is cartesian (Lemma 5.17 shows it is in $\mathscr{M}$).

(2) $\Rightarrow$ (3): immediate

(3) $\Rightarrow$ (1): Given a monic Kleene morphism $m : x \to y$ into a total algebra y, since m is cartesian x is saturated (Proposition 5.13). $\square$

5.20. Remark. The latter equivalence between saturation and ρ_x being a Kleene morphism is Freyd's purported statement that *the envelope construction embeds every paracategory into a category via a Kleene functor* (the set-theoretic proof of this admittedly non-evident statement is unfinished in the draft manuscript [5]). However, as we show in [9], the (composite) adjunction between partial and ordinary algebras is not a useful tool to study the structure of the former; we must keep track of the distinguished subobjects to obtain meaningful results.

An interesting instance of the notion of partial algebra appears in our follow-up article [9]: *partial multicategories*. For a partial multicategory we formulate *repre-*

sentability, in the sense of [6] (for a restricted class of morphisms). A representable partial multicategory provides a suitable axiomatisation for certain 'partial tensor products' which arise in the context of probabilistic automata [15].

Appendix A. Background material

In this section we review the categorical background we use as framework for our treatment of internal paracategories. First, we recall the relevant notions of monoidal category and their morphisms (Section A.1). Secondly, monoidal categories give the ambient structure in which to define monoids. Internal monoids in a monoidal category are classified by (strong) monoidal functors from the simplicial category (Section A.3). Specialising to the setting of endospans in a category with pullbacks, we get the notion internal category as an instance of that of internal monoid (Section A.4).

The context in which we apply (a lax version of) this general theory of internal monoids is that of a bicategory (Section A.2) (or locally ordered category) of partial maps (Section A.5).

A.1. Monoidal categories and their morphisms

Our basic framework is that of a *monoidal category*, i.e. a category $\mathbb{V}$ endowed with a pseudo-associative $\otimes : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ and pseudo-unitary $I \in \mathbb{V}$ structure. See [14,12] for details. The primary examples of interest to us are:

1. $\mathscr{S}et$, the category of sets and functions, with monoidal structure given by cartesian products,
2. $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$, the category of partial maps relative to a class of *monic domains* $\mathscr{M}$ in a category with finite limits $\mathbb{B}$ (which we recall in Section A.5 below), with monoidal structure given by finite products, and
3. $\mathbf{Spn}(\mathbb{B})(C, C)$, the category of endospans on an object C in a category $\mathbb{B}$ with pullbacks, cf. Section A.4.

We recall the relevant notions for morphisms of monoidal categories:

- Given monoidal categories $(\mathbb{M}, \otimes, I)$ and $(\mathbb{N}, \otimes', I')$, a *monoidal functor* between them is a functor $F : \mathbb{M} \to \mathbb{N}$ together with *structural natural transformations*

  $$\gamma : I' \to FI \quad \delta_{x,y} : Fx \otimes' Fy \to F(x \otimes y)$$

  for x, y in $\mathbb{M}$ subject to coherence axioms, which guarantee that we get well-defined comparison transformations $\delta_{\vec{x}} : \otimes_1^n Fx_i \to F(\otimes_1^n x_i)'$ for any n-ary tensor of $\vec{x} = \langle x_1, \ldots, x_n \rangle$.
- A monoidal functor is called *normal* when $\gamma = id : I' \to FI$. It is called *strong* (resp. *strict*) when the structural natural transformations are isomorphisms (resp. identities).
- Given monoidal functors (F, γ, δ) and (F', γ', δ') a *monoidal transformation* between them is a natural transformations $\alpha : F \Rightarrow F'$ compatible with the structural

transformations, i.e.

$$\alpha_I\gamma = \gamma', \quad \alpha_{x\otimes y}\delta_{x,y} = \delta'_{x,y}(\alpha_x \otimes \alpha_y).$$

A.1. Remark. A good way to understand laxity for monoidal categories is via *multicategories*, which yield an effective and simple description of classifiers for lax functors, cf. [6, Remark 9.5]. The monoid classifier Δ is obtained as a special case of this construction.

A.2. Bicategories and lax functors

The notions of monoidal functor and monoidal transformation have their several-objects counterparts in the context of bicategories, where they become *lax functors* and *lax transformations* respectively.

The notion of bicategory is part of the standard categorical tool-kit [14, Section XII.6]. A bicategory $\mathscr{K}$ has objects, morphisms and 2-cells, so that the morphisms $\mathscr{K}(X,Y)$ together with their 2-cells form a category, and there are identities $id_X \in \mathscr{K}(X,Y)$ and composition functors $\otimes : \mathscr{K}(X,Y)\times\mathscr{K}(Y,Z)\to\mathscr{K}(X,Z)$ together with unit $\rho_f : f \Rightarrow \otimes(f,id_X)$, $\lambda_f : \otimes(id_Y,f)\Rightarrow f$ (for $f:X\to Y$) and associativity $\alpha_{f,g,h} : \otimes(\otimes(f,g),h)\Rightarrow\otimes(f,\otimes(g,h))$.

2-cell isomorphisms (for composable f,g,h) satisfying coherence conditions as per a monoidal category.

A *lax functor* $F:\mathscr{K}\to\mathscr{L}$ maps objects to objects, morphisms to morphisms and 2-cells to 2-cells, preserving the source–target relationships. But it is not required to preserve the composition and identities. Instead, we have *comparison* 2-cells $\gamma_X : id_F X \Rightarrow Fid_X$ and $\delta_{f,g} : Ff\circ Fg \Rightarrow F(f\circ g)$ satisfying the evident coherence axioms (like those for monoidal functors). When the comparison 2-cells are isomorphisms, we refer to the lax functor as a *homomorphism of bicategories*.

A *lax transformation* $\alpha : F\Rightarrow F' : \mathscr{K}\to\mathscr{L}$ between lax functors assigns to every morphism $h:X\to Y$ in $\mathscr{K}$ a 2-cell as displayed below

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Fh"'] \arrow[dr, phantom, "\overset{\alpha_h}{\Rightarrow}"] & F'X \arrow[d, "F'h"] \\ FY \arrow[r, "\alpha_Y"'] & F'Y \end{tikzcd}$$

subject to coherence conditions which make them compatible with the comparison 2-cells of F and F'. More information on bicategorical matters (as relevant for this paper) appears in [6], which provides further references for the interested reader.

The two bicategories of interest in this paper are: that of partial maps (Section A.5 below) and that of spans. Given a category $\mathbb{B}$ with pullbacks, we build the *bicategory of spans* **Spn**($\mathbb{B}$): its objects are those of $\mathbb{B}$, morphisms are given by spans $X \overset{d_R}{\leftarrow} R \overset{c_R}{\rightarrow} Y$ and 2-cells $f:R\Rightarrow S$ correspond to arrows between the top objects of the spans such that the following diagram commutes:

$$
\begin{tikzcd}
 & R \arrow[dl, "d_R"'] \arrow[dr, "c_R"] \arrow[dd, "f"] & \\
X & & Y \\
 & S \arrow[ul, "d_S"] \arrow[ur, "c_S"'] &
\end{tikzcd}
$$

The identity morphism on X is the span $X \xleftarrow{id} X \xrightarrow{id} X$ and composition is given by pullback:

$$
\begin{tikzcd}
 & R \arrow[dl, "d_R"'] \arrow[dr, "c_R"] & & & S \arrow[dl, "d_S"'] \arrow[dr, "c_S"] & \\
X & & Y & Y & & Z
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & R\bullet S \arrow[dl, "\overline{d_S}"'] \arrow[dr, "\overline{c_R}"] & & \\
 & R \arrow[dl, "d_R"'] \arrow[dr, "c_R"] & & S \arrow[dl, "d_S"'] \arrow[dr, "c_S"] & \\
X & & Y & & Z
\end{tikzcd}
$$

A.3. Internal monoids and their classifier

A monoidal category provides the ambient structure to define an *internal monoid*, so that for instance an internal monoid in $\mathscr{S}et$ is a monoid in the usual sense, while an internal monoid in Section A.1(3) above amounts to an *internal category*.

We recall from [14, Chapter VI] the basic notion of monoid in a monoidal category and its classifier.

A.2. Definition. Given a monoidal category $\langle \mathbb{C}, \otimes, I, \alpha, \rho, \lambda \rangle$, a *monoid* in it consists of an (*underlying*) object M and morphisms $e : I \to M$ (*unit*) and $m : M \otimes M \to M$ (*multiplication*) satisfying the equations

$$m \circ (e \otimes id_M) \circ \lambda_M = id_M$$

$$m \circ (id_M \otimes e) \circ \rho_M = id_M$$

$$m \circ (m \otimes id_M) = m \circ (id_M \otimes m) \circ \alpha_{M,M,M}.$$

A *monoid morphism* is a morphism between the underlying objects which commutes with unit and multiplication. We thus have the category $\mathscr{M}on(\mathbb{C})$ of monoids in $\mathbb{C}$.

Given monoidal categories $\langle \mathbb{C}, \otimes, I \rangle$ and $\langle \mathbb{C}', \otimes', I' \rangle$, a monoidal functor $F : \mathbb{C} \to \mathbb{C}'$ (with structural cells γ and δ) induces a functor $\mathscr{M}on(F) : \mathscr{M}on(\mathbb{C}) \to \mathscr{M}on(\mathbb{C}')$ as follows:

$$\begin{tikzcd}
I \arrow[r, "e"] & M \arrow[d, mapsto] & & M \otimes M \arrow[ll, "m"'] \\
I' \arrow[r, "\gamma"'] & FI \arrow[r, "Fe"'] & FM & F(M \otimes M) \arrow[l, "Fm"'] & FM \otimes FM \arrow[l, "\delta_{M,M}"']
\end{tikzcd}$$

Furthermore, notice that a monoid in $\mathbb{C}$ amounts to a monoidal functor $M : \mathbf{1} \to \mathbb{C}$ and a monoid morphism to a monoidal transformation between the corresponding monoidal functors. Thus the action of $\mathscr{M}on(F)$ above amounts to composition with F. For a monoidal category $\mathbb{C}$ there is a universal construction of a monoidal category $G(\mathbb{C})$ such that monoidal functors out of $\mathbb{C}$ correspond to strong monoidal functors out of $G(\mathbb{C})$. Here we proceed to recall the explicit description of $G(\mathbf{1})$, which classifies monoids cf. [14, Section VI.5].

A.3. Definition (Simplicial category)**.** Let Δ denote the category of finite ordinals

$$[n] = \{0, \ldots, n-1\}$$

(including the empty one $\emptyset = [0]$) and monotone functions between them. It carries a strict monoidal structure given by ordinal addition $+ : \Delta \times \Delta \to \Delta$, with $\emptyset$ as unit. Since $[1] = \mathbf{1}$ is the terminal object, there are unique arrows $e : \emptyset \to \mathbf{1}$ and $m : \mathbf{1} + \mathbf{1} \to \mathbf{1}$, which satisfy (trivially) the equations for a monoid in Δ.

A.4. Proposition (Classification of monoids)**.** *Given a monoidal category* $\mathbb{C}$, *the functor* $\_(\mathbf{1}) : \mathscr{M}on\mathscr{C}at(\mathbf{\Delta}, \mathbb{C}) \to \mathscr{M}on(\mathbb{C})$ *which evaluates a strong functor* (*resp. transformation*) *at the monoid* $\mathbf{1}$, *induces an equivalence of categories*:

$$\mathscr{M}on\mathscr{C}at(\Delta, \mathbb{C}) \simeq \mathscr{M}on(\mathbb{C}).$$

The above proposition means that all the objects in Δ are finite tensor powers of $\mathbf{1}$, i.e. $[n] = \otimes^n \mathbf{1}$ and that all (non-identity) maps are generated from e and m by $+$ and composition. Thus Δ gives us a neat way to organise all the n-ary operations present in a monoid, as well as their equational theory.

A.4. Internal categories

Given a category $\mathbb{B}$ with finite limits, we can define an *internal category* or a *category object* in it. The most suitable way to phrase such definition for our purposes is via the associated bicategory of spans (Section A.2).

A.5. Definition. An *internal category* C in $\mathbb{B}$ is a monad in $\mathbf{Spn}(\mathbb{B})$, i.e. an object C_0 in $\mathbb{B}$ and a span $C_0 \xleftarrow{d} C_1 \xrightarrow{c} C_0$ endowed with a monoid structure in $\mathbf{Spn}(\mathbb{B})(C_0, C_0)$, given by 2-cells $\iota : C_0 \Rightarrow C_1$ (*identities*) and $m : C_1 \circ C_1 \Rightarrow C_1$ (*composition*) satisfying associativity and unit laws.

An *internal functor* from C to D consists of a pair of morphisms ($f_0 : C_0 \to D_0$, $f_1 : C_1 \to D_1$), commuting with domain, codomain, identity and multiplication maps.

With the evident pointwise composition of internal functors we obtain the category $\mathscr{C}at(\mathbb{B})$ of internal categories and internal functors in $\mathbb{B}$.

A monoid in $\mathbf{Spn}(\mathbb{B})(C_0, C_0)$ corresponds to a strong monoidal functor $C : \Delta \to \mathbf{Spn}(\mathbb{B})(C_0, C_0)$.

A.6. Remark. An internal category in $\mathbb{B}$ amounts also to a finite-limit preserving functor $C : (\Delta_+)^{op} \to \mathbb{B}$, where Δ_+ is the category of non-empty finite ordinals and monotone functions. This combinatorial/topological point of view of a category, due to Grothendieck, is usually referred to as its *nerve*. It is a (special kind of) *simplicial object* in $\mathbb{B}$. But this alternative formulation would be more difficult to generalise to the context of partial maps to obtain paracategories, as we do here.

A.5. Monoidal bicategory of partial maps

Given a category $\mathbb{B}$, we consider a class of monos $\mathscr{M}$ in it such that:

- isomorphisms are in $\mathscr{M}$
- for $m : P \hookrightarrow X$ in $\mathscr{M}$ and $h : Y \to X$ in $\mathbb{B}$, the pullback

$$\begin{tikzcd} f^*P \arrow[r, "\bar{f}"] \arrow[d, hook, "f^*m"'] & P \arrow[d, hook, "m"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

  exists and $f^*m : f^*P \hookrightarrow Y$ belongs to $\mathscr{M}$
- $\mathscr{M}$ is closed under composition.

Such a class $\mathscr{M}$ is called a *dominion* in [18,17], which are standard references for this subject. We call *$\mathscr{M}$-subobject* an isomorphism class of monos in $\mathscr{M}$ (over their common codomain). We abuse notation and write $m : P \hookrightarrow X$ for the equivalence class of m qua $\mathscr{M}$-subobject.

A.7. Examples.

- Consider $\mathbb{B} = \omega\text{-}CPO$ of partial orders with suprema of countable chains and monotone maps preserving them. A suitable class $\mathscr{M}$ of monos is given by the *admissible* subobjects, which are those closed under suprema in the ambient cpo (these are used as they admit Scott's fixpoint induction principle). This is the traditional basic setting of domain theory.
- Consider $\mathbb{B} = \mathscr{S}et^{\mathbb{C}^{op}}$ any presheaf topos. Any topology on it determines suitable classes of $\mathscr{M}$, e.g. the closed subobjects.

A.8. Definition. The bicategory of *partial maps* $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$ consists of

objects those of $\mathbb{B}$

morphisms a morphism $(m, f) : X \rightharpoonup Y$ is given by a span, so that $m : P \hookrightarrow X$ is an $\mathscr{M}$-subobject and $f : P \to Y$ is a morphism in $\mathbb{B}$ (the total part of the partial map). We also write $h : X \rightharpoonup Y$, in which case its total part is the morphism $h : P_h \to Y$, with $d_h : P_h \hookrightarrow X$ the corresponding $\mathscr{M}$-subobject.

2-*cells* Given $(m, f), (n, g) : X \rightharpoonup Y$, a 2-cell between them is a morphism of the corresponding spans. Since there is at most one such (because we are considering $\mathscr{M}$-subobjects), we have in fact a partial order on $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})(X, Y)$.

Composition and identities are inherited from $\mathbf{Spn}(\mathbb{B})$; the conditions on $\mathscr{M}$ ensure that the composition of two partial maps qua spans yields a partial map.

Notice that we have an embedding of $\mathbb{B}$ into $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$:

$$f : X \to Y \mapsto (id, f) : X \rightharpoonup Y.$$

This embedding enjoys a universal property which characterises $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$ up to equivalence [8]. In general, a partial map $(m, f) : X \rightharpoonup Y$ is in the image of this embedding iff $m : P \hookrightarrow X$ is an isomorphism. In this case, $(m, f) : X \rightharpoonup Y$ is called a *total map*. Notice that total maps are *maximal* with respect to the partial order in their hom-sets.

For a more vernacular notation using elements, the expression

$$f(x_1, \ldots, x_n) \geqslant g(x_1, \ldots, x_n)$$

means that $f \leqslant g : M_1 \times \cdots \times M_n \rightharpoonup M$, while

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$$

means (Kleene) equality of partial maps (equal domains and values). With these notation, composition of morphisms amounts to substitution of terms for variables. We also use the *is defined* $(\_)\downarrow$ predicate, so that $f(e)\downarrow$ means that e is in the domain of definition of the partial map f.

A.9. Remark. Notice that the partial order in the hom-sets $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})(X, Y)$ is the $\geqslant$ relation (inclusion of domains and equality of results over the smaller one), while the induced equality is Kleene equality.

We have the following cancellation property of total maps:

A.10. Lemma. *Given partial maps $(m, f) : X \rightharpoonup Y$ and $(n, g) : Y \rightharpoonup Z$, if the composite $(n, g) \circ (m, f)$ is total, then (m, f) is total as well.*

Proof. Consider the composite of spans $(n, g) \circ (m, f)$:

$$\begin{tikzcd}[column sep=small]
 & & f^*Q \circ Q \arrow[dl, hook', "f^*n"'] \arrow[dr, "\overline{f}"] & & \\
 & P \arrow[dl, hook', "m"'] \arrow[dr, "f"] & & Q \arrow[dl, hook', "n"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

By hypothesis, the composite $m \circ f^* n$ is an isomorphism. Hence m is a split epi, and thus an isomorphism. □

The stability conditions for the monos in $\mathscr{M}$ ensure that the finite product structure on $\mathbb{B}$ extends to a monoidal structure on $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$, by applying the product functor to the spans corresponding to the partial maps:

$$\begin{array}{c}
\begin{tikzcd}[column sep=small]
 & P \arrow[dl, hook', "m"'] \arrow[dr, "f"] & & & Q \arrow[dl, hook', "n"'] \arrow[dr, "g"] & \\
X & & Y & U & & V
\end{tikzcd} \\ \hline
\begin{tikzcd}
 & P \times Q \arrow[dl, hook', "m \times n"'] \arrow[dr, "f \times g"] & \\
X \times U & & Y \times V
\end{tikzcd}
\end{array}$$

Furthermore, this monoidal structure is compatible with the partial order on the hom-sets. Thus $\mathsf{Ptl}_{\mathscr{M}}(\mathbb{B})$ is a *monoidal bicategory*, that is a bicategory (actually a 2-category) endowed with a monoid structure with respect to the cartesian product of bicategories.[6]

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[6] There is a more general notion of monoidal bicategory in the literature, where the monoidal structure is understood with respect to the Gray-tensor product of bicategories.

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